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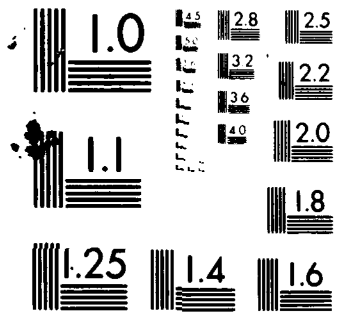


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# THE ASYMPTOTIC BEHAVIOR OF A FREE BOUNDARY ARISING FROM A BISTABLE REACTION-DIFFUSION EQUATION

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# THE ASYMPTOTIC BEHAVIOR OF A FREE BOUNDARY ARISING FROM A BISTABLE REACTION-DIFFUSION EQUATION

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## ABSTRACT

The pure initial value problem for the bistable reaction-diffusion equation

$$v_t = v_{xx} + f(v)$$

is considered. Here $f(v)$ is given by $f(v) = -v + H(v-a)$, and $a \in (0,1/2)$. Of primary interest is the asymptotic behavior of the curve $s(t)$ given by $s(t) = \sup\{x : v(x,t) = a\}$. It is shown that there exist a unique constant $c$ such that if the initial datum is greater than the parameter $a$ on a sufficiently long interval, then $\lim_{t\to\infty} (s(t) - ct)$ exists.

AMS (MOS) Subject Classification: 35K55

Key Words: Reaction-diffusion equation, Traveling wave solution

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## SIGNIFICANCE AND EXPLANATION

The mathematical equation studied here has been considered as a model for a variety of physical phenomena including population genetics and nerve conduction. Of primary interest is the eventual behavior of solutions of this equation. One expects that for sufficiently large initial datum the solutions should eventually look like some type of wave traveling with constant shape and velocity. In the case of nerve conduction, for example, the initial datum may correspond to a stimulus applied to the nerve axon. Physiologically, it has been demonstrated that if this stimulus is greater than some threshold amount, then a signal will propagate down the axon with a speed independent of the initial stimulus. In this paper we demonstrate that the equation under study supports solutions exhibiting similar behaviors.



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THE ASYMPTOTIC BEHAVIOR OF A FREE BOUNDARY
ARISING FROM A BISTABLE REACTION-DIFFUSION EQUATION

David Terman

## Section 1. Introduction

The bistable reaction-diffusion equation

$$v_t = v_{xx} + f(v) \quad , \tag{1.1}$$

with $f$ as qualitively shown in Figure 1, has been studied by a number of authors as a qualitative model for population genetics, combustion, and nerve conduction (see [1], [2]).

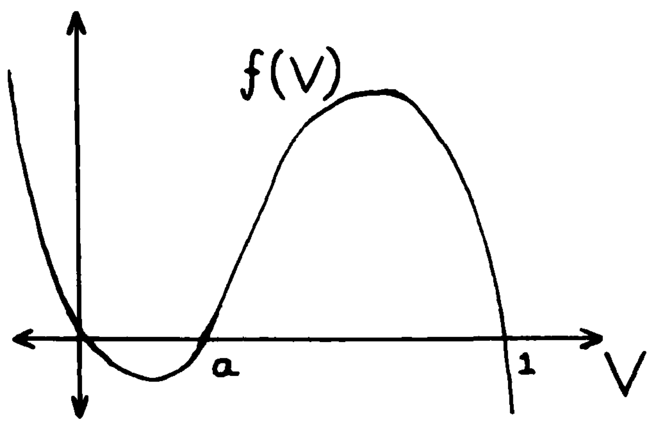


Figure 1

By a traveling wave solution of (1.1) we mean a nonconstant, bounded solution of the form $v(x,t) = v_c(z)$, $z = x-ct$. Note that these correspond to solutions traveling with constant shape and velocity. It has been demonstrated that equation (1.1) possesses a unique traveling wave satisfying $\lim_{z\to-\infty} v_c(z) = 0$ and $\lim_{z\to\infty} v_c(z) = 1$. (see Aronson and Weinberger [1]). It has also been demonstrated that equation (1.1) exhibits a threshold phenomenon. Aronson and Weinberger [1] considered the initial value problem for (1.1) and showed that if the initial datum, $v(x,0)$, is sufficiently small, then

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$\lim_{t\to\infty} \|v(\cdot,t)\|_{L^\infty} = 0$. In this case we say that the initial datum is subthreshold. If, however, $v(x,0)$ is sufficiently large, or superthreshold, then $\lim_{t\to\infty} v(x,t) = 1$ for all $x \in R$. Fife and McLeod [2] showed that in the case when the initial datum is superthreshold, the solution asymptoticly approaches a traveling wave solution.

In this paper we consider the pure initial value problem for equation (1.1) with $f(v)$ given by the piecewise continuous function

$$f(v) = -v + H(v-a) \quad .$$

Here $H$ is the Heaviside step function and $a \in (0,1/2)$. As before, there exists a unique traveling wave solution with a unique speed $c^*$. In fact, Rinzel and Keller [3] showed that

$$c^* = (1-2a)[a(1-a)]^{-1/2} \quad .$$

In [4] it is proved that if the initial datum, $v(x,0) = \varphi(x)$, satisfies the conditions

$$(1.2) \qquad \begin{array}{lll} \text{(a)} & \varphi(x) \in C^2(\mathbb{R}) \quad , & \\ \text{(b)} & \varphi(x) \in [0,1] & \text{in } \mathbb{R} \quad , \\ \text{(c)} & \varphi(x) = \varphi(-x) & \text{in } \mathbb{R} \quad , \\ \text{(d)} & \varphi'(x) < 0 & \text{in } \mathbb{R}^+ \quad , \\ \text{(e)} & \varphi(x_0) = a & \text{for some } x_0 > 0 \quad , \end{array}$$

with $x_0$ sufficiently large, then the curve $s(t)$, given by

$$(1.2c) \qquad s(t) = \sup\{x : v(x,t) = a\}$$

is a well defined smooth function such that $\lim_{t\to\infty} s(t) = \infty$. Note that in some sense $x_0$ determines the size of the initial datum. From the results of Fife and McLeod one expects more to be true. That is, in the superthreshold

case the solution should asymptoticlly propagate with constant shape and velocity. In this paper we prove the following result which indicates that the solution eventually propagates with constant velocity.

Theorem 1.1: There exists a constant $\theta$ such that if $\varphi(x)$ satisfies (1.2) with $x_0 > \theta$ then $\lim_{t\to\infty} [s(t) - c^*t]$ exists.

Actually, in the proof of this theorem we do not need the assumption (1.2c). All that is required is that the curve $s(t)$ is continuous in $\mathbb{R}^+$, $\lim_{t\to\infty} s(t) = \infty$, and $v(x,t) > a$ if and only if $|x| < s(t)$.

In [4] it is shown that the curve $s(t)$ must satisfy the integral equation

$$(1.3) \quad a - \int_{-\infty}^{\infty} K(s(t) - \xi,\ t)\varphi(\xi)d\xi = \int_0^t \int_{-s(\tau)}^{s(\tau)} K(s(t) - \xi,\ t - \tau)d\xi d\tau$$

where

$$(1.4) \qquad K(x,t) = \frac{e^{-t}}{2\pi^{1/2}t^{1/2}} e^{-\frac{x^2}{4t}}$$

is the fundamental solution of the linear differential equation:

$$(1.5) \qquad \psi_t = \psi_{xx} - \psi \quad .$$

One can see, intuitively, that this is true for the following reason.

Let $G_T = \{(x,t) : |x| < s(t),\ 0 < t < T\}$ and let $\chi_{G_T}$ be the indicator function of the region $G_T$. Since $v > a$ if and only if $|x| < s(t)$ it follows that for $|x| \neq s(t)$, $v(x,t)$ satisfies the inhomogeneous equation

$$(1.6) \qquad \begin{aligned} v_t &= v_{xx} - v + \chi_{G_T} && \text{in } \mathbb{R} \times (0,T) \ , \\ v(x,0) &= \varphi(x) && \text{in } \mathbb{R} \ . \end{aligned}$$

Formally the solution of (1.5) can be written as

$$(1.7) \quad v(x,t) = \int_{-\infty}^{\infty} K(x - \xi,\ t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} K(x - \xi,\ t - \tau)\chi_{G_T} d\xi d\tau \quad .$$

Setting $x = s(t)$ in (1.7) one obtains (1.3). Theorem 1.1 is proved by analyzing the integral equation (1.3).

We now introduce some notation. Let

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x - \xi, t)\varphi(\xi)d\xi \quad .$$

Note that $\psi(x,t)$ is the solution of the linear differential equation (1.5) with initial datum $\psi(x,0) = \varphi(x)$. Furthermore, since $\varphi(x) \in [0,1]$ in $\mathbb{R}$ it follows from the maximum principle that $0 < \psi(x,t) < e^{-t}$ in $\mathbb{R}^+$.

Suppose that $\alpha(t)$ is a positive, continuous function defined in $[0,T]$. For $t \in [0,T]$, let

$$\Phi(\alpha)(t) = \int_0^t \int_{-\alpha(\tau)}^{\alpha(\tau)} K(\alpha(t) - \xi, t - \tau)d\xi d\tau$$

$$\Theta(\alpha)(t) = a - \psi(\alpha(t), t) \quad .$$

Note that $s(t)$ is a solution of the integral equation (1.3) in $(0,T)$ if and only if $\Theta(s)(t) = \Phi(s)(t)$ in $(0,T)$.

The following lemma is an immediate consequence of the definition of $\Phi$. We state it here because it is used so often in the proof of Theorem 1.1.

Lemma 1.2. Assume that $\alpha(t) \leq \beta(t)$ in $[0,T]$ and $\alpha(T) = \beta(T)$. Then $\Phi(\alpha)(T) \leq \Phi(\beta)(T)$. If $\alpha(t) < \beta(t)$ for some $t \in (0,T)$, then $\Phi(\alpha)(T) < \Phi(\beta)(T)$.

## Section 2: Preliminary Results

Recall that $c^*$ is the velocity of the unique traveling wave solution of equation (1.1). In this paper it will be useful to think about $c^*$ in a different fashion. Suppose that $u(x,t) = u_{c^*}(x - c^*t)$ is the unique traveling wave solution of equation (1.1), and let $\sigma(t)$ be defined implicitly as $u(\sigma(t), t) = a$. Since the translate of a traveling wave is also a traveling wave we may assume that $\sigma(0) = 0$. Then $\sigma(t)$ is given explicitly as $\sigma(t) = c^*t$. A derivation similar to that given for equation (1.3) shows that $\sigma(t)$ must satisfy the integral equation:

$$a - \int_{-\infty}^{\infty} K(\sigma(t) - \xi, t)u_{c^*}(\xi)d\xi = \int_0^t \int_{-\infty}^{\sigma(\tau)} K(\sigma(t) - \xi, t - \tau)d\xi d\tau \quad .$$

Using the change in variables $\eta = \tau - t$, $\zeta = \xi - c^*t$ in the integral on the right hand side of this equation, we find that

$$(2.1) \qquad a - \int_{-\infty}^{\infty} K(c^*t - \xi, t)u_{c^*}(\xi)d\xi = \int_{-t}^{0} \int_{-\infty}^{c^*\eta} K(-\zeta,-\eta)d\zeta d\eta$$

for each $t \in R^+$. Letting $t \to \infty$ in (2.1) we find that $c^*$ must satisfy the equation:

$$(2.2) \qquad a = \int_{-\infty}^{0} \int_{-\infty}^{c^*\tau} K(-\xi, -\tau)d\xi d\tau \quad .$$

To see that there must exist a unique solution, $c^*$, of equation (2.2) we let $h(c)$ be the function defined by

$$(2.3) \qquad h(c) = \int_{-\infty}^{0} \int_{-\infty}^{c\tau} K(-\xi,-\tau)d\xi d\tau \quad .$$

It is not hard to see that $h(0) = 1/2$, $h'(c) < 0$, and $\lim_{c\uparrow\infty} h(c) = 0$. Since $a \in (0, 1/2)$ there must exist a unique solution of (2.2).

We conclude this section by showing that for $x_0$ sufficiently large, both $\liminf_{t\to\infty} (s(t) - c^*t)$ and $\limsup_{t\to\infty} (s(t) - c^*t)$ exist. This is done by constructing continuous functions $\alpha(t)$ and $\beta(t)$ which satisfy $\alpha(t) < s(t) < \beta(t)$, and both $\lim_{t\to\infty} (\alpha(t) - c^*t)$ and $\lim_{t\to\infty} (\beta(t) - c^*t)$ exist. The construction of $\alpha(t)$ goes as follows.

Let $M = -h'(c^*)$. Then $M > 0$, and, since $h(c^*) = a$, there exists a positive constant $\varepsilon$ such that $\varepsilon < \frac{c^*}{2}$, and if $|\delta| < \varepsilon$, then $h(c^* - \delta) > a + \frac{\delta M}{2}$.

For $(x,t) \in R \times R^+$, let

(2.4) $$B(t) = \int_{-\infty}^{0} \int_{-\infty}^{\infty} K(x - \xi, t - \tau)d\xi d\tau = e^{-t}$$

and

(2.5) $$g(x,t) = \int_0^t \int_{-\infty}^{0} K(x - \xi, t - \tau)d\xi d\tau \quad .$$

Note that if $(x,t) \in (1,\infty) \times R^+$, then

(2.6) $$g(x,t) < 2e^{-x} \quad .$$

This is because, if $(x,t) \in (1,\infty) \times R^+$, then

$$g(x,t) = \int_0^t \int_{-\infty}^{0} \frac{e^{-(t-\tau)}}{2\pi^{1/2}(t-\tau)^{1/2}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} d\xi d\tau$$

$$< \int_0^t \int_{-\infty}^{0} \frac{e^{-(t-\tau)}}{2\pi^{1/2}(t-\tau)^{1/2}} e^{-\frac{(x-\xi)}{4(t-\tau)}} d\xi d\tau$$

$$= \frac{2}{\pi^{1/2}} e^{-x} \int_0^t (t-\tau)^{1/2} e^{-(t-\tau)} d\tau$$

$$< 2e^{-x} \quad .$$

Before we construct $\alpha(t)$, it is necessary to define a few more constants. Let $r = \min\{c^* - \varepsilon, 1\}$. Choose $N$ so large that $\frac{3}{M} e^{-rN} < \varepsilon$, and let

(2.7) $$\lambda = c^*N + \frac{6}{Mr} e^{-rN} \quad .$$

Finally, choose $\theta$ so large that if $x_0 > \theta$, then $s(t) > \lambda$ in $R^+$. This is possible because of Theorem 6.4 of [4].

Let

$$\alpha(t) = \begin{cases} \lambda & \text{for } t \in [0,N) \\ c^*t + \frac{3}{Mr}[e^{-rt} + e^{-rN}] & \text{for } t \geq N \quad . \end{cases}$$

Note that $\alpha(t)$ is a continuous function and

$$\alpha'(t) = c^* - \frac{3}{M} e^{-rt} \tag{2.8}$$

for $t > N$. Hence, $\alpha'(t)$ is increasing for $t > N$, and $\lim_{t\to\infty} \alpha'(t) = c^*$.

We now show that $\alpha(t) < s(t)$ in $\mathbb{R}^+$. Clearly this is true for $t \in [0,N]$. To prove that $\alpha(t) < s(t)$ for $t > N$ we show that $\Phi(\alpha)(t) > a$ for $t > N$. This will imply the desired result for the following reason. Suppose that $\alpha(T) = s(T)$ for some $T > N$. We assume that $\alpha(t) < s(t)$ for $t < T$. Then, from Lemma 1.2, $\Phi(s)(t) > \Phi(\alpha)(T) > a$. However, $\Theta(s)(T) = a - \psi(s(T),T) < a$. Since $\Phi(s)(T) = \Theta(s)(T)$ this is impossible.

So suppose that $T > N$. We wish to show that $\Phi(\alpha)(T) > a$. Let $\ell(t)$ be the line tangent to $\alpha(t)$ at $t = T$. That is, $\ell(t) = \alpha'(T)(t-T) + \alpha(T)$. Since $\alpha''(t) > 0$ for $t > N$, it follows that $\alpha(t) > \ell(t)$ in $(0,T)$. It follows from Lemma 1.2 that $\Phi(\alpha)(T) > \Phi(\ell)(T)$. We prove that $\Phi(\ell)(T) > a$.

Note that $\ell(0) > 0$. This because,

$$\begin{aligned}\ell(0) &= -\alpha'(T)T + \alpha(T) \\ &= -[c^* - \frac{3}{M} e^{-rT}]T + c^*T + \frac{3}{Mr} [e^{-rT} + e^{-rN}] \\ &> 0 \;.\end{aligned}$$

This implies that

$$\begin{aligned}\Phi(\ell)(T) &= \int_{-\infty}^{T} \int_{-\infty}^{\ell(\tau)} K(\ell(T) - \xi, T - \tau)d\xi d\tau \\ &\quad - \int_{-\infty}^{0} \int_{-\infty}^{\ell(\tau)} K(\ell(T) - \xi, T - \tau)d\xi d\tau - \int_{0}^{T} \int_{-\infty}^{-\ell(\tau)} K(\ell(T) - \xi, T - \tau)d\xi d\tau \\ &> h(\alpha'(T)) - B(T) - q(\ell(T), T) \;.\end{aligned}$$

Now, $h(\alpha'(T)) = h(c^* - \frac{3}{M} e^{-rT})$. Since $\frac{3}{M} e^{-rN} < \varepsilon$, and $T > N$, it follows that $h(\alpha'(T)) > a + 3e^{-rT}$. On the other hand, $B(T) = e^{-T} < e^{-rT}$. To

estimate $q(\alpha(T),T)$, note that, since $\ell(0) > 0$, it follows that $\alpha(T) > \alpha'(T)T$. Hence, (2.6) implies that

$$q(\ell(T),T) < 2e^{-\alpha'(T)T)} \leq 2e^{-(c^*-\epsilon)T} \leq 2e^{-rT}.$$

These comments prove that $\Phi(\ell)(T) > a$ and, hence, $\Phi(\alpha)(T) > a$. This completes the construction of $\alpha(t)$, and the proof that $\alpha(t)$ has the desired properties.

We now construct $\beta(t)$. Let $t_n = -\log\frac{a}{2n}$, $n = 1,2,\ldots,$ and choose $c_n$ so that $h(c_n) = a - \frac{a}{2n}$. Note that $c_n \downarrow c^*$ as $n \to \infty$. Let $\lambda_1 = 2 \sup\limits_{0<t<t_1} s(t)$. For $t \in [0,t_1]$ let $\beta(t) = c_1 t + \lambda_1$, and for $t > t_1$ let $\beta(t)$ to be the continuous, piecewise-linear function defined by $\beta'(t_n) = c_n$ for $t \in (t_n, t_{n+1})$. Clearly $\beta(t)$ is a well defined function which satisfies $\beta'(t) \to c^*$ as $t \to \infty$. It remains to prove that $s(t) < \beta(t)$ in $R^+$.

Certainly $s(t) < \beta(t)$ in $[0,t_1]$. Suppose there exists $T > t_1$ such that $s(T) = \beta(T)$ and $s(t) < \beta(t)$ for $t < T$. Assume that $T \in [t_n, t_{n+1}]$. We show that this must imply that

$$\Theta(s)(T) > a - \frac{a}{2n} > \Phi(s)(T)$$

which contradicts the fact that $s(t)$ is a solution of (1.3).

First of all, note that $\psi(s(T), T) < e^{-T}$. In fact, $\psi(x,t) < e^{-t}$ for all $(x,t) \in R^+ \times R^+$. This follows from the maximum principle applied to (1.5) and our assumption that $\psi(x,0) \in [0,1]$. Since $T \in [t_n, t_{n+})$ we have that $\psi(s(T), T) < e^{-t_n} = \frac{a}{2n}$. It now follows from the definition of $\Theta$ that $\Theta(s)(T) > a - \frac{a}{2n}$.

It remains to prove that $\Phi(s)(T) < a - \frac{a}{2n}$. Let $\ell(t)$ be the line defined by

$$\ell(t) = c_n(t-T) + s(T) \quad .$$

Then $s(t) \leq \beta(t) \leq \ell(t)$ in $(0,T)$. From Lemma 1.2, it follows that $\Phi(s)(T) \leq \Phi(\ell)(T)$. However,

$$\Phi(\ell)(T) < \int_{-\infty}^{T} \int_{-\infty}^{\ell(\tau)} K(\ell(T) - \xi,\ T - \tau)d\xi d\tau$$

$$= h(c_n) \quad .$$

Since $h(c_n) = a - \frac{a}{2n}$, we obtain the desired contradiction.

## Section 3: Proof of Theorem 1.1

Before continuing with the proof of Theorem 1.1 we introduce some notation which will be used throughout the rest of this paper.

Let $\lambda = \limsup_{t\to\infty} (s(t) - c^*)$. Choose $\{t_n\}$, $n = 1,2,\ldots,$ so that,

(3.1a) $$s(t_n) > c^*t + \lambda - 1/n$$

and

(3.1b) $$s(t) < c^*t + \lambda + 1/n \quad \text{for} \quad t > t_n \quad .$$

Let

$$\ell_n(t) = c^*(t - t_n) + s(t_n)$$

$$J_n = \{t < t_n : \ell_n(t) < s(t)\}$$

$$H_n = \{t < t_n : s(t) \leq \ell_n(t)\}$$

$$A_n = \int_{J_n} \int_{\ell_n(\tau)}^{s(\tau)} K(s(t_n) - \xi,\ t_n - \tau)d\xi$$

$$B_n = \int_{H_n} \int_{s(\tau)}^{\ell_n(\tau)} K(s(t_n) - \xi,\ t_n - \tau)d\xi \quad .$$

Note that (3.1) implies that $s(t) < \ell_n(t) + \frac{2}{n}$ if $t > t_n$.

The next couple of lemmas give us some sort of estimate of how much the curve $s(t)$ can oscillate for $t < t_n$, $n = 1,2,\ldots$ . They demonstrate that for $n$ large, and $t < t_n$, the curve $s(t)$ must be very close to the line $\ell_n(t)$ in some sort of weighted $L^1$ sense.

<u>Lemma 3.1</u>: $A_n \to 0$ as $n \to \infty$.

**Proof:** Fix $m < n$. Since $s(t) < \ell_m(t) + \frac{2}{m}$ for $t > t_m$, it follows that

$$A_n < \int_{-\infty}^{t_m} \int_{-\infty}^{\infty} K(s(t_n)-\xi,\ t_n - \tau)d\xi d\tau + \int_{t_m}^{t_n} \int_{\ell_n(\tau)}^{\ell_m(\tau)+\frac{2}{m}} K(s(t_n)-\xi, t_n-\tau)d\xi d\tau$$

$$= [I] + [II] \quad .$$

Now,

$$[I] < \int_{-\infty}^{t_m} e^{-(t_n-\tau)} d\tau < e^{-(t_n-t_m)} \quad .$$

On the other hand,

$$[II] < \int_{-\infty}^{t_n} \int_{\ell_n(\tau)}^{\ell_n(\tau)+\frac{4}{m}} K(s(t_n) - \xi,\ t_n - \tau)d\xi d\tau$$

$$= \int_0^{4/m} \int_{-\infty}^{0} K(-\eta - c^*\tau,\ -\tau)d\tau d\eta$$

$$< \frac{M}{m}$$

for some constant $M$ which does not depend on $m$ and $n$. We have shown that

$$A_n < \frac{M}{m} + e^{-(t_n-t_m)}$$

for all $m < n$. Let $m = \frac{n}{2}$ if $n$ is even and $m = \frac{n+1}{2}$ if $n$ is odd. It follows that $A_n < \frac{2M}{n} + e^{-\frac{t_n}{2}}$, and the proof of the lemma is complete.

**Lemma 3.2:** $B_n \to 0$ as $n \to \infty$.

**Proof:** Note that

$$\Phi(s)(t_n) = \int_{-\infty}^{t_n} \int_{-\infty}^{\ell_n(\tau)} K(s(t_n)-\xi, t_n-\tau)d\xi d\tau - \int_{-\infty}^{0} \int_{-\infty}^{\ell_n(\tau)} K(s(t_n)-\xi, t_n-\tau)d\xi d\tau$$

$$+ \int_0^{t_n} \int_{\ell_n(\tau)}^{s(\tau)} K(s(t_n)-\xi, t_n-\tau)d\xi d\tau - \int_0^{t_n} \int_{-\infty}^{s(\tau)} K(s(t_n)-\xi, t_n-\tau)d\xi d\tau$$

$$\equiv [I] - [II] + [III] - [IV] < [I] + [III] \quad .$$

Now, $[I] = h(c^*) = a$, while, $[III] = A_n - B_n$. Since

$$\Phi(s)(t_n) = \Theta(s)(t_n) = a - \psi(s(t_n), t_n)$$

it follows that

$$B_n \leq A_n + \psi(s(t_n), t_n) \quad .$$

Since each term on the right hand side of this equation $\to 0$ as $n \to \infty$, the result follows.

Here we give a brief outline of how the proof of Theorem 1.1 will be completed. For each $n$ we construct a sequence of positive constants $\{\delta_{nk}\}$, $k = 0,1,\ldots,$ with the property that if $\delta_n = \sum_{k=0}^{\infty} \delta_{nk}$, then $\delta_n \to 0$ as $n \to \infty$. Furthermore, letting $h_n(t)$ be the piecewise continuous function defined by

$$(3.2) \qquad h_n(t) = \begin{cases} s(t) & \text{for} \quad t \leq t_n \\ \ell_n(t) - \sum_{j=0}^{k} \delta_{nj} & \text{for} \quad t_n + k < t \leq t_n + k + 1 \end{cases} ,$$

we show that $h_n(t) \leq s(t)$ for each $n$. This implies that $\ell_n(t) - \delta_n < s(t)$ for each $n$ and $t > t_n$. Since we already know that $s(t) < \ell_n(t) + \frac{2}{n}$ for each $n$ and $t > t_n$, this will complete the proof of Theorem 1.1. In what follows we set $t_{nk} \equiv t_n + k$.

The $\delta_{nk}$ are defined inductively. Fix $n$ and suppose we have already chosen $\delta_{n1},\ldots,\delta_{n,k-1}$. Furthermore assume that for $t < t_{nk}$, $h_n(t) < s(t)$ where $h_n(t)$ is defined by (3.2). We show how to define $\delta_{nk}$. It must be

chosen in such a way that $h_n(t) < s(t)$ for $t \in (t_{nk}, t_{n,k+1}]$. From the definition of $\delta_{nk}$ it will be clear that $\delta_n \equiv \sum_{k=0}^{\infty} \delta_{nk} \to 0$ as $n \to \infty$.

<u>Lemma 3.3</u>: If $\delta_{nk}$ is sufficiently large then $\Phi(h_{nk})(t) > a$ $t \in (t_{nk}, t_{n,k+1}]$.

Before proving the lemma we show that it implies that $h_n(t) < s(t)$ in $(t_{nk}, t_{n,k+1}]$. If this were not true, then there must exist some $T \in (t_{nk}, t_{n,k+1}]$ such that $h_n(T) = s(T)$, and $h_n(t) < s(t)$ for all $t < T$. From Lemma 1.2 this would imply that $\Phi(s)(T) > \Phi(h_n)(T) > a$. Since $\Theta(s)(t) < a$ for all $t$ this is a contradiction.

<u>Proof of Lemma 3.3.</u>

Assume that $t \in (t_{nk}, t_{n,k+1}]$. To simplify the notation we set $h_{nj}(t) = \ell_n(t) - \sum_{i=0}^{j} \delta_{ni}$ for $t \in \mathbf{R}$. Then (3.2) becomes

$$h_n(t) = \begin{cases} s(t) & \text{for} \quad t < t_n \\ h_{nj}(t) & \text{for} \quad t \in (t_{nj}, t_{n,j+1}] \end{cases} .$$

Note that,

$$\Phi(h_n)(t) = \int_{-\infty}^{t} \int_{-\infty}^{h_{nk}(\tau)} K(h_n(t) - \xi,\ t - \tau)d\xi d\tau$$

$$- \left[ \int_{-\infty}^{0} \int_{-\infty}^{h_{nk}(\tau)} K(h_n(t)-\xi, t-\tau)d\xi d\tau + \int_{0}^{t} \int_{-\infty}^{-h_n(\tau)} K(h_n(t)-\xi, t-\tau)d\xi d\tau \right]$$

$$+ \left[ \int_{0}^{t} \int_{h_{nk}(\tau)}^{h_n(\tau)} K(h_n(t) - \xi,\ t - \tau)d\xi d\tau \right]$$

$$= a - [\mathrm{I}] + [\mathrm{II}] \quad .$$

Now, using the notation of the previous section,

$$[I] \le B(t) + q(h_n(t),t) \le e^{-t} + 2e^{-c^*t} \le 3e^{-rk}e^{-rt_n} \quad .$$

Here we set $r = \min(1,c^*)$. Next consider [II]. Using the fact that $h_n(\tau) = h_{nk}(\tau)$ for $\tau \in (t_{nk},t)$, we rewrite [II] as

$$[II] = \int_0^{t_{nk}} \int_{h_{nk}(\tau)}^{h_{n,k-1}(\tau)} K(h_n(t)-\xi,t-\tau)d\xi d\tau - \int_0^{t_{nk}} \int_{h_n(\tau)}^{h_{n,k-1}(\tau)} K(h_n(t)-\xi,t-\tau)d\xi d\tau$$

$$= [II_1] - [II_2] \quad .$$

Note that

$$[II_1] \ge \int_{t-2}^{t-1} \int_{h_{nk}(\tau)}^{h_{n,k-1}(\tau)} K(h_n(t) - \xi,\ t - \tau)d\xi d\tau$$

$$= \int_0^{\delta_{nk}} \int_{-2}^{-1} K(-c^*t - \eta,\ -\tau)d\tau d\eta$$

$$\ge \delta_{nk}M_1$$

for some constant $M_1$ which does not depend on $n$. On the other hand, since $h_n(\tau) = s(\tau)$ for $\tau < t_n$, $h_{n,k-1}(\tau) < h_n(\tau)$ for $\tau \in (t_n,t_{n,k-1})$, and $h_{n,k-1}(\tau) < \ell_n(\tau)$ for $\tau < t_n$,

$$[II_2] < \int_0^{t_n} \int_{h_n(\tau)}^{h_{n,k-1}(\tau)} K(h_n(t)-\xi,t-\tau)d\xi d\tau$$

$$< \int_{H_n} \int_{s(\tau)}^{\ell_n(\tau)} K(h_n(t) - \xi,\ t - \tau)d\xi d\tau$$

$$= \int_{H_n\cap(0,t_n-1)} \int_{s(\tau)}^{\ell_n(\tau)} K(h_n(t)-\xi,t-\tau)d\xi d\tau + \int_{H_n\cap(t_n-1,t_n)} \int_{s(\tau)}^{\ell_n(\tau)} K(h_n(t)-\xi,t-\tau)d\xi d\tau$$

$$= [II_{21}] + [II_{22}] \quad .$$

To estimate $[II_{21}]$ we set $G_n = \{(\xi,\tau) : \tau \in H_n \cap (0,t_{n-1}),\ \ s(\tau) < \xi < \ell_n(\tau)\}$,

and $P_n(t) = \sup\limits_{(\xi,\tau)\in G_n} \dfrac{K(h_n(t)-\xi,t-\tau)}{K(s(t_n)-\xi,t_n-\tau)}$. Then

$$[II_{21}] = \int_{G_n}\int K(s(t_n)-\xi,t_n-\tau)\,\frac{K(h_n(t)-\xi,t-\tau)}{K(s(t_n)-\xi,t_n-\tau)}\,d\xi d\tau$$

$$\leq P_n(t)\,B_n\ .$$

Using (1.4) one finds that there exists a constant $M_2$, independent of $n$, such that if $t \in (t_{nk},t_{n,k+1})$, then $P_n(t) \leq e^{-k}M_2$. Hence,

$$[II_{21}] \leq e^{-rk}M_2B_n\ .$$

Finally consider $[II_{22}]$. Let $\mathcal{M}_n$ = measure $H_n \cap (t_{n-1},t_n)$. Note that $\mathcal{M}_n \to 0$ as $n \to \infty$. This follows because $B_n \to 0$ as $n \to \infty$. Hence,

$$[II_{22}] \leq \int_{H_n\cap(t_{n-1},t_n)} \int_{-\infty}^{\infty} K(h_n(t)-\xi,t-\tau)d\xi$$

$$\leq \int_{H_n\cap(t_{n-1},t_n)} e^{-(t-\tau)}d\tau \leq \int_{t_n-\mathcal{M}_n}^{t_n} e^{-(t-\tau)}d\tau$$

$$\leq e^{-k}[1 - e^{-\mathcal{M}_n}]\ .$$

Setting $\overline{M}_n = 1 - \mathcal{M}_n$, we have that $[I_{22}] \leq \overline{M}_n e^{-rk}$ . Note that $\overline{M}_n \to 0$ as $n \to \infty$.

Combining all of these estimates, we have shown that

$$\Phi(h_n)(t) > a - 3e^{-rk}e^{-rt_n} + \delta_{nk}M_1 - e^{rk}[M_2B_n + \overline{M}_n]\ .$$

Hence $\Phi(h_n)(t) > a$ if we set

$$\delta_{nk} = \frac{e^{-rk}}{M_1}[3e^{-rt_n} + M_2B_n + \overline{M}_n] \equiv K_ne^{-rk}\ .$$

Note that $K_n \to 0$ as $n \to \infty$. An immediate consequence is that if $\delta_n = \sum\limits_{k=0}^{\infty} \delta_{nk}$, then $\delta_n \to 0$ as $n \to \infty$.

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The pure initial value problem for the bistable reaction-diffusion equation

$$v_t = v_{xx} + f(v)$$

is considered. Here $f(v)$ is given by $f(v) = -v + H(v-a)$, and $a \in (0,1/2)$. Of primary interest is the asymptotic behavior of the curve $s(t)$ given by $s(t) = \sup\{x : v(x,t) = a\}$. It is shown that there exist a unique constant $c$ such that if the initial datum is greater than the parameter $a$ on a sufficiently long interval, then $\lim_{t\to\infty} (s(t) - ct)$ exists.

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